AN ANALYSIS OF SECOND-ORDER SLIP FLOW AND TEMPERATURE-JUMP BOUNDARY CONDITIONS FOR RAREFIED GASES

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Abstract—The effects of second-order terms on the velocity and temperature jumps at a wall are obtained by a physical derivation. The analysis uses the concepts of effective mean free paths for momentum and energy transfer; the effective mean free paths are obtained from known viscosities and thermal conductivities. The second-order slip flow analysis is applicable at somewhat lower pressures than is the first-order analysis and applies to non-monatomic as well as to monatomic gases. Several illustrative examples, including fully developed flow and heat transfer in a tube are considered. Differences between the first- and second-order corrections on the order of 20 per cent were noted in the region for which the analysis appears applicable.

NOMENCLATURE

A,   area (Fig. 1);

a,   accommodation coefficient [equation (40)];

c_p,   specific heat at constant pressure;

c_v,   specific heat at constant volume;

E,   energy crossing dA per unit area per unit time from a given direction;

E^+,   total energy crossing dA per unit area per unit time from above;

E^-,   total energy crossing dA per unit area per unit time from below;

F,   fraction of molecules reflected diffusely [equation (19)];

f,   Maxwellian distribution function for molecular speeds [equation (6)];

I,   internal energy per unit mass other than translational energy;

k,   thermal conductivity;

l,   molecular free path;

$l_{e,i}$,   effective free path for internal energy transfer;

$l_{e,m}$,   effective free path for momentum transfer;

$l_{e,t}$,   effective free path for translational energy transfer;

l_h,   hard-sphere free path,

M_x, x-component of momentum crossing dA per unit time per unit area from a given direction;

M_x^+,   total x-momentum crossing dA per unit area per unit time from above;

M_x^-,   total x-momentum crossing dA per unit area per unit time from below;

m,   molecular mass;

Nu,   Nusselt number, $2q_0 r_0/(T_w - T_b)k$;

Nu_c,   continuum Nusselt number;

n,   number density,

Pr,   Prandtl number, $c_p \mu/k$;

p,   pressure;

Δp,   actual pressure drop;

Δp_c,   pressure drop for continuum flow at velocity at which actual pressure drop is Δp;

q,   heat transfer per unit area per unit time;

R,   gas constant;

r,   tube radius;

T,   temperature;

U_x, x-component of mean velocity;

u_x, x-component of molecular velocity;

v,   molecular speed;

v_x,   portion of x-velocity component that is random [equation (3)];

x, y, z,   co-ordinates (Fig. 1).

Greek symbols

α,   thermal diffusivity;

γ, c_p/c_v;

θ,   spherical co-ordinate (Fig. 1);

$\kappa,$	Boltzmann's constant;
$\mu,$	viscosity;
$\rho,$	density;
$\tau,$	shear stress;
$\varphi,$	spherical co-ordinate (Fig. 1);
$\Omega,$	defined by equation (14);
$\omega,$	angular velocity of dA;

Subscripts

$b,$	bulk;
$i,$	referring to internal energy other than translational;
$m,$	monatomic;
$r,$	reflected;
$t,$	referring to translational energy;
$v,$	in velocity range dv at velocity v;
$w,$	wall;
$0,$	referring to point x_0, y_0, z_0 (Fig. 1) or to gas at wall;
$1, 2,$	referring to planes 1 and 2;

Superscript

| $^-,$ | overbar designates mean value. |

INTRODUCTION

SLIP and temperature-jump boundary conditions have been used with considerable success in the analysis of slightly rarefied gases [1]. In this method of analysis, the continuum equations of momentum and energy are used throughout the gas, and the effects of the walls are taken into account by using appropriate boundary conditions. For a rarefied gas with velocity and temperature gradients, the velocity and temperature of the gas next to the wall will differ from those of the wall. The gas next to the wall is made up of molecules coming from the wall and from a distance a mean free path away from the wall, so that its velocity and temperature will be between those of the wall and of the gas a mean free path away. If the mean free path is small, the velocity and temperature jumps will be negligible.

In the usual analysis, the velocity and temperature jumps at the wall are assumed to be proportional to the normal velocity and temperature gradients. That is a good assumption if the velocity and temperature profiles are essentially linear over a mean free path, as they will be if the gas is but slightly rarefied. At somewhat

lower pressures, however, where the profiles may be non-linear over a mean free path, the jumps at the wall would be expected to be functions of the higher order normal and tangential derivatives.

Second-order jump boundary conditions have been obtained in reference [2] by using Burnett's approximate solution of the Boltzmann equation. Burnett's equations, however, have been found to give results that are not always in agreement with experiment [1]; in fact, in many cases the Navier–Stokes equations were found to be superior. Thus, attempting to obtain second-order jump boundary conditions by using a comparatively simple physical derivation* appears to be worthwhile. The Boltzmann equation will not be used herein, but the momentum and energy carried across an area element by molecules that, in effect, had their last collision a distance equal to an effective mean free path from the element will be considered. The effective mean free path, which has different values for momentum and heat transfer and which also differs from the usual hard-sphere mean free path, is then related to experimental viscosities and thermal conductivities. The results differ somewhat from those of reference 2.

The expressions for the velocity jump at a wall will be derived in the next section, after which the corresponding temperature jump will be considered. The results, which use Eucken's approximation [4], are applicable to both monatomic and non-monatomic gases. Interactions between the velocity and temperature fields, such as thermal creep, are neglected.

MOMENTUM TRANSFER

Consider the x-component of momentum carried by molecules across an area element dA located at x_0, y_0, z_0. The plane of dA is normal to the z-axis (Fig. 1). If all the particles were traveling in a direction making an angle θ with the z-axis and a polar angle φ, the number of particles in the velocity range between v and $v + dv$ that pass though dA per second would be $n_0 f_0 \, dv \, v \cos \theta \, dA$. The quantity n_0 is the number density of particles at x_0, y_0, z_0, and f_0

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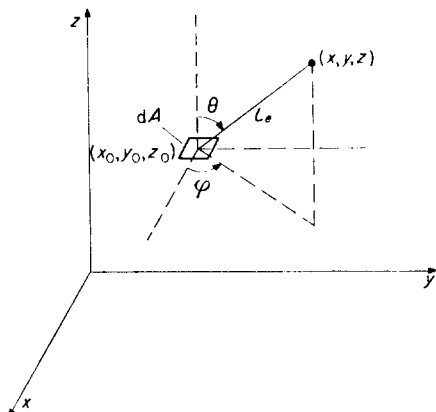* A related analysis for thermal radiation in gas was recently carried out by the author (3).

is the corresponding velocity distribution function. For an isotropic gas, the fraction of particles with velocities that make an angle between θ and $\theta + d\theta$ with the z-axis and a polar angle between φ and $\varphi + d\varphi$ is $\sin \theta \, d\theta \, d\varphi/4\pi$. Thus, the actual number of particles in the velocity range dv that pass through dA per unit time at an angle with the z-axis between θ and $\theta + d\theta$ and a polar angle between φ and $\varphi + d\varphi$ is

$$dZ_v = n_0 f_0 \, dv \, v \cos \theta \, dA \sin \theta \, d\theta \, d\varphi/4\pi \qquad (1)$$

To carry out the analysis, assume first, that the mean velocity of the stream is uniform and in the x-direction. (The effect of velocity gradients will be considered subsequently.) Then the x-momentum carried across dA by molecules that are in the velocity range dv and move at an angle to the z-axis between θ and $\theta + d\theta$ and at a polar angle between φ and $\varphi + d\varphi$ is

$$dM_{x,v} = mu_{x,0} \, n_0 f_0 \, dv \, v \cos \theta \, dA \sin \theta \, d\theta \, d\varphi/4\pi \qquad (2)$$

where m is the molecular mass and $u_{x,0}$ is the x-component of the velocity in the range dv.

Let

$$u_{x,0} = U_{x,0} + v_{x,0} \qquad (3)$$

where $U_{x,0}$ is the mean x-component of velocity at 0, and $v_{x,0}$ is the portion of the x-component of velocity that is random. The random portion will have a Maxwellian velocity distribution. By use of equation (3), equation (2) can be integrated over all velocities to give

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$$dM_x = dA \, (\sin \theta \cos \theta \, d\theta \, d\varphi/4\pi) \int_0^\infty mu_{x,0} \, n_0 f_0 \, dv$$
$$= dA \, (\sin \theta \cos \theta \, d\theta \, d\varphi/4\pi) \, mn_0 \times$$
$$(U_{x,0} \int_0^\infty v f_0 \, dv + \sin \theta \cos \varphi \int_0^\infty v^2 f_0 \, dv) \qquad (4)$$

where v_x is written in spherical co-ordinates as

$$v_x = v \sin \theta \cos \varphi \qquad (5)$$

and

$$f_0 = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{\kappa T_0}\right)^{3/2} v^2 \exp\left(-\frac{(1/2) \, mv^2}{\kappa T_0}\right) \qquad (6)$$

Equation (6) gives, of course, the Maxwellian distribution function for molecular speeds. Equation (4) becomes

$$dM_x = [dA \, (\sin \theta \cos \theta \, d\theta \, d\varphi/4\pi) \, n_0 \bar{v}_0] \, [mU_{x,0}$$
$$+ (3/8) \, \pi \, m\bar{v}_0 \sin \theta \cos \varphi] \qquad (7)$$

The last term in this equation is obtained by use of the relation

$$\overline{v^2} = (3/8) \, \pi \, \bar{v}^2$$

which can be obtained by using equation (6). Equation (7) gives the x-momentum transferred per unit time across dA by molecules whose velocities make an angle between θ and $\theta + d\theta$ with the z-axis and a polar angle between φ and $\varphi + d\varphi$, if the gas is moving at uniform velocity.

If equation (1) is integrated over all values of v, the quantity in the first brackets in equation (7) is obtained. Thus that quantity gives the number of molecules that cross dA from the given angle range per unit time, and the quantity in the second brackets can be interpreted as the effective x-momentum carried per molecule. If the mass velocity is not uniform, the molecules will carry momentum that differs from $mU_{x,0} + (3/8) \, \pi m\bar{v}_0 \sin \theta \cos \varphi$. Molecules that, in effect, had their last collision a distance $l_{e,m}$ (effective mean free path for momentum transfer) from dA will carry x-momentum equal to $mU_{x,l} + (3/8) \, \pi m\bar{v}_l \sin \theta \cos \varphi$. Thus, equation (7) becomes

$$dM_x = [dA \, (\sin \theta \cos \theta \, d\theta \, d\varphi/4\pi) \, n_0 \bar{v}_0] \, [mU_{x,l} +$$
$$(3/8) \, \pi \, m\bar{v}_l \sin \theta \cos \varphi] \qquad (8)$$

The mass velocity $U_{x,l}$ at a point x,y,z (Fig. 1) can be related to conditions at x_0, y_0, z_0 by

expanding U_x in a three-dimensional Taylor series about x_0, y_0, z_0. This expansion gives

$$U_{x,l} = \sum_{h=0}^{\infty} \frac{1}{h!} \left[(z - z_0) \left(\frac{\partial}{\partial z} \right)_0 + (y - y_0) \left(\frac{\partial}{\partial y} \right)_0 + (x - x_0) \left(\frac{\partial}{\partial x} \right)_0 \right]^h U_x$$

If the binomial theorem is applied twice to the factor in brackets,

$$U_{x,l} = \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{(z - z_0)^{h-w} (y - y_0)^{w-s} (x - x_0)^s}{(h - w)! \, (w - s)! \, s!} \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 \tag{9}$$

Equation (9) can be written in spherical co-ordinates $l_{e,m}$, θ, φ with origin at dA by setting

$$x - x_0 = l_{e,m} \sin \theta \cos \varphi, \, y - y_0 = l_{e,m} \sin \theta \sin \varphi, \, z - z_0 = l_{e,m} \cos \theta$$

It should be emphasized that $l_{e,m}$ will, in general, be greater than the distance to the actual point of the last collision because of the persistence of velocities. That is, after a collision many of the molecules tend to continue traveling in the direction they traveled before collision. Equation (9) becomes, in spherical co-ordinates,

$$U_{x,l} = \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{l_{e,m}^h \cos^{h-w} \theta \sin^w \theta \sin^{w-s} \varphi \cos^s \varphi}{(h - w)! \, (w - s)! \, s!} \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 \tag{10}$$

In equation (8), the term $(3/8) \, \pi m \bar{v}_l \sin \theta \cos \varphi$ gives the contribution of the random molecular velocities to the momentum transfer. If temperature gradients in the flow direction are assumed to be small, that term will drop out when we integrate over direction to get the total x-momentum passing through dA from above. If thermal gradients in the flow direction are large, that term may produce thermal creep effects, but those effects are neglected here. Thus, for simplicity, the last term in equation (8) will be omitted in the remainder of the analysis.

Substituting equation (10) in equation (8) and averaging over all values of $l_{e,m}$ give

$$\mathrm{d}M = \frac{m n_0 \bar{v}_0 \, \mathrm{d}A}{4\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{\overline{l^h}_{e,m} \sin^{w+1} \theta \cos^{h-w+1} \theta \sin^{w-s} \varphi \cos^s \varphi}{(h - w)! \, (w - s)! \, s!} \, \mathrm{d}\theta \, \mathrm{d}\varphi \times \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 \tag{11}$$

where the overbar on $\overline{l^h}_{e,m}$ signifies an averaged value. To calculate $\overline{l^h}_{e,m}$ in terms of $\bar{l}_{e,m}$, the distribution function ψ for molecular free paths must be known. Jeans [5] has shown that ψ is given approximately by

$$\psi = \frac{1}{c\bar{l}} \exp \left(-\frac{l}{c\bar{l}} \right)$$

where c is a constant on the order of one, which accounts for the fact that l varies with velocity. Thus,

$$\overline{l^h} = \frac{1}{c\bar{l}} \int_0^{\infty} l^h \exp \left(-\frac{l}{c\bar{l}} \right) \mathrm{d}l = h! \, (c\bar{l})^h$$

This form is also assumed to apply to $l_{e,m}$. Thus,

$$\overline{l^h}_{e,m} = h! \, \bar{l}^h_{e,m} \tag{12}$$

where the constant c has been absorbed in the value of $\bar{l}_{e,m}$, ($\bar{l}_{e,m}$ will later be related to known viscosities). Substituting equation (12) in (11) and integrating to obtain the total x-momentum passing through dA from above results in

$$dM_x^+ = \frac{mn_0\bar{v}_0 \, dA}{4\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{\bar{l}_{e,m}^h h!}{(h-w)! \, (w-s)! \, s!} \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s}\right)_0 \times \int_0^{\pi/2} \int_0^{\pi 2} \sin^{w+1}\theta$$

$$\cos^{h-w+1}\theta \, \sin^{w-s}\varphi \, d\varphi \, d\theta = \frac{\rho_0 v_0 \, dA}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \Omega \, (h. \, w, \, s) \, \bar{l}_{e,m}^h \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s}\right)_0 \qquad (13)$$

where

$$\Omega \, (h, \, w, \, s) = \frac{[1+(-1)^{w-s}][1+(-1)^s] \, h! \, \Gamma[(h-w+2)/2] \, \Gamma[(w-s+1)/2] \, \Gamma[(s+1)/2]}{(h-w)! \, (w-s)! \, s! \, \Gamma[(h+4)/2]} \qquad (14)$$

and mn_0 is replaced by the mass density ρ_0. The symbol Γ stands for the gamma function. In order to obtain the x-momentum passing through dA from below, we let θ go from $\pi/2$ to π, instead of from 0 to $\pi/2$, and change the sign of the result:

$$dM_x^- = \frac{\rho_0\bar{v}_0 \, dA}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} (-1)^{h-w} \Omega \, (h, \, w, \, s) \, \bar{l}_{e,m}^h \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s}\right)_0. \qquad (15)$$

The shear stress acting on dA is the net x-momentum transferred per unit area through dA from above:

$$\tau_0 = \frac{dM_x^+ - dM_x^-}{dA}$$

or

$$\tau_0 = \frac{\rho_0\bar{v}_0}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} [1-(-1)^{h-w}] \, \Omega \, (h, \, w, \, s) \, \bar{l}_{e,m}^h \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s}\right)_0 \qquad (16)$$

Next the velocity slip at a tangentially moving wall that is immediately below, but not touching the area dA will be obtained. In order to do this, instead of a wall, a uniform gas below dA moving at the velocity U_w is first considered. Equation (15) then yields for dM_x^-

$$dM_x^- = \frac{\rho_0\bar{v}_0 \, dA}{16\pi} \Omega \, (0, 0, 0) \, U_w = \frac{1}{4} \rho_0\bar{v}_0 \, dA \, U_w \qquad (17)$$

If the term for $h = 0$ is extracted from the summation in equation (13) and equation (17) is subtracted from that equation, there results

$$dM_x^+ - dM_x^- = \frac{1}{4} \rho_0\bar{v}_0 \, dA \, (U_0 - U_w) + \frac{\rho_0\bar{v}_0 \, dA}{16\pi} \sum_{h=1}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \Omega \, (h, \, w, \, s) \, \bar{l}_{e,m}^h \left(\frac{\partial^h U_x}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s}\right)_0.$$

$$(18)$$

Consider next a wall moving at the velocity U_w, rather than a uniform gas, to be below dA. Then the fraction of the momentum of the molecules relative to the wall, which is, on the average, given up to the wall, is

$$F = \frac{dM_x^+ - dM_{x,r}}{dM_x^+ - dM_x^-} \qquad (19)$$

where $dM_{x,r}$ is the momentum carried by reflected molecules. The momentum difference in the denominator of equation (19) is for a wall with perfect momentum accommodation and is taken to be the same as that which occurs when a uniform moving gas is below dA. The quantity F is sometimes interpreted as the fraction of molecules reflected diffusely, the rest being reflected specularly. The quantity $dM_x^+ - dM_{x,r}$, which is the net x-momentum transferred through dA from above, is dA times the shear stress. The shear stress is given by equation (16), since that equation is assumed to apply throughout the gas; the effect of the wall is accounted for by the jump boundary conditions. Setting $\tau_0\, dA = dM_x^+ - dM_{x,r}$ in equation (16), substituting that equation and equation (18) in (19), and solving for $U_0 - U_w$ result in

$$U_0 - U_w = \frac{1}{4\pi} \sum_{h=1}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \left[\frac{1 - (-1)^{h-w} - F}{F} \right] \Omega\,(h, w, s)\, l_{e,m}^h \left(\frac{\partial^h U_x}{\partial z^{h-w}\, \partial y^{w-s}\, \partial x^s} \right)_0. \tag{20}$$

If we retain only terms through second order (terms containing second derivatives), equation (20) becomes

$$U_0 - U_w = \frac{2}{3} \frac{(2 - F)}{F} l_{e,m} \left(\frac{\partial U_x}{\partial z} \right)_0 - \frac{1}{2} l_{e,m}^2 \left[\left(\frac{\partial^2 U_x}{\partial z^2} \right)_0 + \frac{1}{2} \left(\frac{\partial^2 U_x}{\partial y^2} \right)_0 + \frac{1}{2} \left(\frac{\partial^2 U_x}{\partial x^2} \right)_0 \right]. \tag{21}$$

Equation (16), correct to terms of second order, is

$$\tau = \frac{1}{3} \rho \bar{v}\, l_{e,m} \frac{\partial U_x}{\partial z} = \mu \frac{\partial U_x}{\partial z} \tag{22}$$

where the subscripts 0 have been dropped because the equation is assumed applicable throughout the gas. Terms containing second derivatives are zero in equation (22). The Navier–Stokes equations can be derived from equation (22) and are thus applicable in the present analysis where second-order boundary conditions are used. It is significant that the Navier–Stokes equations give better results for rarefied gases or for large velocity gradients than certain other approximations, for instance, the Burnett equations [1].

From equation (22),

$$l_{e,m} = 3\mu/(\rho \bar{v})$$

and, since $\rho = p/RT$ for a perfect gas, and $\bar{v} = (8RT/\pi)^{1/2}$ [equation (6)],

$$l_{e,m} = \frac{3}{2} \sqrt{\left(\frac{\pi}{2} \right)} \frac{\mu \sqrt{RT}}{p}. \tag{23}$$

Thus, equation (21) becomes

$$U_0 - U_w = \sqrt{\left(\frac{\pi}{2} \right)} \frac{(2 - F)}{F} \frac{\mu \sqrt{RT}}{p} \left(\frac{\partial U_x}{\partial z} \right)_0 - \frac{9\pi}{16} \left(\frac{\mu \sqrt{RT}}{p} \right)^2 \left[\left(\frac{\partial^2 U_x}{\partial z^2} \right)_0 + \right.$$
$$\left. \frac{1}{2} \left(\frac{\partial^2 U_x}{\partial y^2} \right)_0 + \frac{1}{2} \left(\frac{\partial^2 U_x}{\partial x^2} \right)_0 \right]. \tag{24}$$

Equation (24) is written in terms of measurable quantities. The first term on the right-hand side of equation (24) is the usual first-order slip term [6 (p. 296)], and the second term gives the second-order contributions. Equation (24) applies for a wall below the gas. For a wall above the gas, a similar derivation gives.

$$U_w - U_0 = \sqrt{\left(\frac{\pi}{2} \right)} \frac{(2 - F)}{F} \frac{\mu \sqrt{RT}}{p} \left(\frac{\partial U_x}{\partial z} \right)_0 + \frac{9\pi}{16} \left(\frac{\mu \sqrt{RT}}{p} \right)^2 \left[\left(\frac{\partial^2 U_x}{\partial z^2} \right)_0 + \right.$$
$$\left. \frac{1}{2} \left(\frac{\partial^2 U_x}{\partial y^2} \right)_0 + \frac{1}{2} \left(\frac{\partial^2 U_x}{\partial x^2} \right)_0 \right]. \tag{25}$$

Equations (22), (24), and (25) are the equations for second-order rectilinear slip flow according to the present method of analysis. They differ somewhat from those of reference 2. For instance, the numerical coefficient on $\partial^2 U_x/\partial z^2$ in equation (24) differs from that in reference 2, and the second derivatives with respect to x and y are absent in the corresponding expression in reference 2. It appears, however, from the present physical derivation that those derivatives should have an effect.

Equations (22), (24), and (25) were derived on the assumption that the flow is rectilinear. If the fluid does not move in straight lines, as for concentric rotating cylinders, the area element dA will rotate, and the molecules crossing it will appear to have a different U_x than they would have if the fluid were moving in straight lines. This effect can be taken into account by replacing $\partial U_x/\partial z$ in equations (22), (24), and (25) by $\partial U_x/\partial z + \omega = \partial U_x/\partial z, + \partial U_z/\partial x$, and $\partial^2 U_x/\partial z^2$ by $\partial^2 U_x/\partial z^2 + \partial \omega/\partial z = \partial^2 U_x/\partial z^2 + \partial^2 U_z/\partial x \, \partial z$, where ω is the angular velocity of dA. Equation (22) then becomes

$$\tau = \mu \left(\frac{\partial U_x}{\partial z} + \omega\right) = \mu \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) \tag{22a}$$

which is the generalized expression for shear stress used for deriving the Navier–Stokes equations. Similarly, equation (24) becomes

$$U_0 - U_w = \sqrt{\left(\frac{\pi}{2}\right)} \frac{(2-F)}{F} \frac{\mu\sqrt{RT}}{p} \left[\left(\frac{\partial U_x}{\partial z}\right)_0 + \omega_0\right] - \frac{9\pi}{16}\left(\frac{\mu\sqrt{RT}}{p}\right)^2 \left[\left(\frac{\partial^2 U_x}{\partial z^2}\right)_0 + \right.$$
$$\left. \left(\frac{\partial \omega}{\partial z}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 U_x}{\partial y^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 U_x}{\partial x^2}\right)_0\right]. \tag{24a}$$

ENERGY TRANSFER

The analysis of energy transfer is somewhat analogous to that of momentum transfer in the preceding section. Here, the energy carried by molecules across an area element dA is considered. (See Fig. 1.) The number of particles dZ_v in the velocity range dv that pass through dA per unit time and make an angle with the z-axis between θ and $\theta + d\theta$ and a polar angle between φ and $\varphi + d\varphi$ is again given by equation (1). First, the temperature of the gas is assumed to be uniform. Then, the energy carried across dA by molecules that are in the velocity range dv and move at an angle to the z-axis between θ and $\theta + d\theta$ and at a polar angle between φ and $\varphi + d\varphi$ is

$$dE_v = [(1/2)\,mv^2 + mI_0]\,n_0 f_0\,dv\,v\cos\theta\,dA\sin\theta\,d\theta\,d\phi/(4\pi) \tag{26}$$

where mI_0 is the internal energy of the molecules (energy other than translational). Integration of equation (26) over all molecular speeds with I_0 independent of molecular speed gives

$$dE = dA \left(\frac{\sin\theta\cos\theta\,d\theta\,d\varphi}{4\pi}\right) n_0 \left(\frac{1}{2}m\int_0^\infty v^3 f_0\,dv + mI_0\int_0^\infty v f_0\,dv\right) =$$
$$dA \left(\frac{\sin\theta\cos\theta\,d\theta\,d\varphi}{4\pi}\right) n_0 \left[\frac{1}{2}m(\overline{v^3})_0 + m\bar{v}_0 I_0\right] \tag{27}$$

Using equation (6) for f_0 gives the relation

$$(\overline{v^3})_0 = (4/3)\,(\bar{v})_0\,(\overline{v^2})_0 \tag{28}$$

so that

$$dE = \left[dA \left(\frac{\sin\theta\cos\theta\,d\theta\,d\varphi}{4\pi}\right) n_0\bar{v}_0\right] \left[\frac{4}{3}\left(\frac{1}{2}\right) m\,(\overline{v^2})_0 + mI_0\right]. \tag{29}$$

As in momentum transfer, the quantity in the first bracket gives the number of molecules that cross dA from the given angle range per unit time. The quantity $(4/3)(1/2)m(\overline{v^2})_0$ is the effective translational energy, and mI_0 is the internal energy carried by each molecule. The factor $4/3$ appears in the expression for the effective translational energy because the molecules with large translational kinetic energy cross dA in greater numbers than do the slower moving ones. Equation (29) applies to a gas at a uniform temperature. If the temperature is not uniform, the molecules crossing dA will carry effective kinetic energy equal to $(4/3)(1/2)m(\overline{v^2})_l$, where $(1/2)m(\overline{v^2})_l$ is the average kinetic energy of molecules a distance $l_{e,t}$ from dA and $l_{e,t}$ is the effective l for translational energy transfer. Similarly, the molecules will carry internal energy equal to mI_l, which is the average internal energy of molecules a distance $l_{e,i}$ from dA. The quantity $l_{e,i}$ is the effective l for internal energy transfer and is not necessarily equal to $l_{e,t}$. Thus, equation (29) becomes

$$dE = \left[dA \left(\frac{\sin\theta \cos\theta \, d\theta \, d\varphi}{4\pi} \right) n_0 \bar{v}_0 \right] \left[\frac{4}{3} \left(\frac{1}{2} \right) m \, (\overline{v^2})_l + m I_l \right] \tag{30}$$

Proceeding as for momentum transfer yields, in place of equation (10),

$$(\overline{v^2})_l = \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{l_{e,t}^h \cos^{h-w}\theta \sin^w\theta \sin^{w-s}\varphi \cos^s\varphi}{(h-w)!\,(w-s)!\,s!} \left(\frac{\partial^h \overline{v^2}}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right)_0 \tag{31}$$

and

$$I_l = \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{l_{e,i}^h \cos^{h-w}\theta \sin^w\theta \sin^{w-s}\varphi \cos^s\varphi}{(h-w)!\,(w-s)!\,s!} \left(\frac{\partial^h I}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right)_0 \tag{32}$$

Substituting equations (31) and (32) in equation (30) and averaging over all values of $l_{e,t}$ and $l_{e,i}$ give

$$dE = \left[dA \left(\frac{\sin\theta \cos\theta \, d\theta \, d\varphi}{4\pi} \right) n_0 \bar{v}_0 \right] m \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{\cos^{h-w}\theta \sin^w\theta \sin^{w-s}\varphi \cos^s\varphi}{(h-w)!\,(w-s)!\,s!}$$
$$\left\{ \frac{4}{3} \overline{l_{e,t}^h} \left[\frac{\partial^h (1/2\, \overline{v^2})}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right]_0 + \overline{l_{e,i}^h} \left(\frac{\partial^h I}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right)_0 \right\}. \tag{33}$$

Substituting for $\overline{l_{e,t}^h}$ and $\overline{l_{e,i}^h}$ from equation (12) (with $l_{e,m}$ replaced by $l_{e,t}$ or $l_{e,i}$) and integrating to obtain the total energy passing through dA from above give

$$dE^+ = \frac{m n_0 v_0 \, dA}{4\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{h!}{(h-w)!\,(w-s)!\,s!} \left\{ \frac{4}{3} \overline{l_{e,t}^h} \left[\frac{\partial^h (1/2\,\bar{v}^2)}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right]_0 \right.$$
$$\left. + \overline{l_{e,t}^h} \left(\frac{\partial^h I}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right)_0 \right\} \int_0^{\pi/2} \int_0^{2\pi} \sin^{w+1}\theta \cos^{h-w+1}\theta \sin^{w-s}\varphi \cos^s\varphi \, d\varphi \, d\theta$$

$$= \frac{\rho_0 \bar{v}_0 \, dA}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \Omega\,(h,w,s) \left\{ \frac{4}{3} \overline{l_{e,t}^h} \left[\frac{\partial^h (1/2\,\bar{v}^2)}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right]_0 + \overline{l_{e,i}^h} \left(\frac{\partial^h I}{\partial z^{h-w}\,\partial y^{w-s}\,\partial x^s} \right)_0 \right\} \tag{34}$$

where $\Omega\,(h,w,s)$ is again given by equation (14). The change in total thermal energy of a molecule is $d[(1/2)m\overline{v^2} + mI] = mc_v \, dT$, where c_v is the specific heat at constant volume and T is the temperature. After Eucken [4], c_v is written as $c_{v,t} + c_{v,i}$, so that $d(1/2\,\overline{v^2}) = c_{v,t}\,dT$ and

$dI = c_{v,i} \, dT$. If the variation of $c_{v,i}$ with temperature for derivatives of higher order than the first is neglected, equation (34) then becomes

$$dE^+ = \frac{\rho_0 \bar{v}_0 \, dA}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \Omega\,(h,\,w,\,s) \left(\frac{4}{3} I_{e,t}^h \, c_{v,t} + I_{e,i}^h \, c_{v,i} \right) \left(\frac{\partial^h T}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 \qquad (35)$$

for the energy crossing dA from above. Similarly, the energy crossing dA from below is [equation (15)]

$$dE^- = \frac{\rho_0 \bar{v}_0 \, dA}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} (-1)^{h-w} \Omega\,(h,\,w,\,s) \left(\frac{4}{3} I_{e,t}^h \, c_{v,t} + I_{e,i}^h \, c_{v,i} \right) \left(\frac{\partial^h T}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 . \qquad (36)$$

The net energy or heat transferred in the direction z is

$$q_z = \frac{dE^- - dE^+}{dA}$$

or

$$q_z = -\frac{\rho_0 \bar{v}_0}{16\pi} \sum_{h=0}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} [1 - (-1)^{h-w}] \, \Omega\,(h,\,w,\,s) \left(\frac{4}{3} I_{e,t}^h \, c_{v,t} + I_{e,i}^h \, c_{v,i} \right) \left(\frac{\partial^h T}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 \qquad (37)$$

In order to obtain the effect of a wall on the heat transfer, we assume first, that there is a gas below dA at the uniform temperature T_w. Equation (36) then yields, for dE^-,

$$dE^- = \frac{1}{4} \rho_0 \bar{v}_0 \, dA \left(\frac{4}{3} c_{v,t} + c_{v,i} \right) T_w \qquad (38)$$

If the term for $n = 0$ is extracted from the summation in equation (35) and equation (38) is subtracted from that equation,

$$dE^+ - dE^- = \frac{1}{4} \rho_0 \bar{v}_0 \, dA \left(\frac{4}{3} c_{v,t} + c_{v,i} \right) (T_0 - T_w) + \frac{\rho_0 \bar{v}_0 \, dA}{16\pi} \sum_{h=1}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \Omega\,(h,\,w,\,s) \left(\frac{4}{3} I_{e,t}^h \, c_{v,t} + \right.$$
$$\left. I_{e,i}^h \, c_{v,i} \right) \left(\frac{\partial^h T}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 . \qquad (39)$$

If a wall at temperature T_w, rather than a uniform gas, is placed below dA, the accommodation coefficient a is defined by

$$a = \frac{dE^+ - dE_r}{dE^+ - dE^-} \qquad (40)$$

where dE_r is the energy reflected from the wall. As in the case of F [equation (19)], the accommodation coefficient a is regarded as a quantity to be determined by free-molecule flow experiments, inasmuch as its value depends on many variables and is difficult to predict [7]. But $dE^+ - dE_r$, the net energy transferred through dA in the z-direction, is $-dA$ times the heat transfer per unit area and is given by equation (37). Setting $-q_z \, dA = dE^+ - dE_r$ in equation (37), substituting that equation and equation (39) in equation (40), and solving for $T_0 - T_w$ give

$$T_0 - T_w = \frac{1}{4\pi} \sum_{h=1}^{\infty} \sum_{w=0}^{h} \sum_{s=0}^{w} \frac{1 - (-1)^{h-w} - a}{a} \Omega\,(h,\,w,\,s) \frac{[(4/3) I_{e,t}^h \, c_{v,t} + I_{e,i}^h \, c_{v,i}]}{(4/3) \, c_{v,t} + c_{v,i}} \left(\frac{\partial^h T}{\partial z^{h-w} \, \partial y^{w-s} \, \partial x^s} \right)_0 .$$
$$(41)$$

If only terms through second order are retained,

$$T_0 - T_w = \frac{2}{3}\frac{(2-a)}{a}\frac{[(4/3)\,\bar{l}_{e,t}\,c_{v,t} + \bar{l}_{e,i}\,c_{v,i}]}{(4/3)\,c_{v,t} + c_{v,i}}\left(\frac{\partial T}{\partial z}\right)_0 - \frac{1}{2}\frac{[(4/3)\,\bar{l}_{e,t}^2\,c_{v,t} + \bar{l}_{e,i}^2\,c_{v,i}]}{(4/3)\,c_{v,t} + c_{v,i}}$$

$$\times \left[\left(\frac{\partial^2 T}{\partial z^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 T}{\partial y^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_0\right]. \tag{42}$$

The heat transfer, correct through terms of second order, is given by equation (37) as

$$q_z = -\frac{1}{3}\rho\bar{v}\left(\frac{4}{3}\bar{l}_{e,t}\,c_{v,t} + \bar{l}_{e,i}\,c_{v,i}\right)\frac{\partial T}{\partial z} = -k\frac{\partial T}{\partial z} \tag{43}$$

or

$$k = \frac{1}{3}\rho\bar{v}\left(\frac{4}{3}\bar{l}_{e,t}\,c_{v,t} + \bar{l}_{e,i}\,c_{v,i}\right) \tag{44}$$

The subscripts 0 have again been dropped because equation (43) is assumed to apply throughout the gas. For a monatomic gas, equation (44) becomes

$$k_m = \frac{4}{9}\rho\bar{v}\,\bar{l}_{e,t}\,c_{v,t} \tag{45}$$

But k_m is related to μ and $c_{v,t}$ by

$$k_m = \frac{5}{2}\mu c_{v,t} \tag{46}$$

[6 (p. 178)]. From equations (45) and (46),

$$\bar{l}_{e,t} = \frac{45}{8}\frac{\mu}{\rho\bar{v}}$$

or

$$\bar{l}_{e,t} = \frac{45}{16}\sqrt{\left(\frac{\pi}{2}\right)}\frac{\mu\sqrt{RT}}{p} \tag{47}$$

Following Eucken [4], it is assumed that internal energy is transferred in the same way as momentum, so that, by equation (23),

$$\bar{l}_{e,i} = \frac{3}{2}\sqrt{\left(\frac{\pi}{2}\right)}\frac{\mu\sqrt{RT}}{p} \tag{48}$$

Eucken also assumed that $c_{v,t} = (3/2)R$, that is, that the transfer of translational energy is unaffected by the presence of internal energy. Since $c_v = c_{v,t} + c_{v,i}$ and $c_v = R/(\gamma-1)$,

$$c_{v,t} = \frac{3\,(\gamma-1)}{2}c_v, \; c_{v,i} = \frac{(5-3\gamma)}{2}c_v \tag{49}$$

Equations (44), (47), (48), and (49) give

$$k = \frac{1}{4}(9\gamma - 5)\,\mu c_v \tag{50}$$

which is Eucken's formula and has been found to give results for most gases that are in good agreement with experiment [6 (p. 180)]. Substitution of equations (44), (47), (48), and (49) in equation (42) gives

$$T_0 - T_w = \sqrt{(2\pi)}\frac{(2-a)}{a}\frac{\gamma}{(\gamma+1)\,Pr}\frac{\mu\sqrt{RT}}{p}\left(\frac{\partial T}{\partial z}\right)_0 - \frac{9\pi}{256}\frac{(177\gamma - 145)}{\gamma+1}\left(\frac{\mu\sqrt{RT}}{p}\right)^2$$

$$\times \left[\left(\frac{\partial^2 T}{\partial z^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 T}{\partial y^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_0\right] \tag{51}$$

for a wall below the gas. A similar derivation gives

$$T_w - T_0 = \sqrt{(2\pi)}\frac{(2-a)}{a}\frac{\gamma}{(\gamma+1)\,Pr}\frac{\mu\sqrt{RT}}{p}\left(\frac{\partial T}{\partial z}\right)_0 + \frac{9\pi}{256}\frac{(177\gamma - 145)}{\gamma+1}\left(\frac{\mu\sqrt{RT}}{p}\right)^2$$

$$\times \left[\left(\frac{\partial^2 T}{\partial z^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 T}{\partial y^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_0\right] \tag{52}$$

for a wall above the gas. Equations (43), (51), and (52) are second-order equations for heat transfer in a rarefied gas according to the present method of analysis. The application of these equations and those in the preceding section to several problems will be given in the following sections.

PLANE COUETTE FLOW AND HEAT TRANSFER

For plane Couette flow with frictional heating neglected, the shear stress and heat transfer are independent of the distance from a wall. Thus equations (22) and (43) become

$$U_x - U_1 = \tau z/\mu \qquad (53)$$

and

$$T - T_1 = -\frac{q_z z}{k} \qquad (54)$$

Using velocity and temperature jumps at the two walls calculated from equations (24), (25) and (51) to (54) yields

$$\frac{(U_{w,2} - U_{w,1})\mu}{\tau L} = \sqrt{(2\pi)}\frac{(2-F)}{F}\frac{\mu\sqrt{RT}}{pL} + 1 \qquad (55)$$

and

$$\frac{(T_{w,1} - T_{w,2})k}{q_z L} = 2\sqrt{(2\pi)}\frac{(2-a)}{a}\frac{\gamma}{(\gamma+1)\,Pr}$$
$$\left(\frac{\mu\sqrt{RT}}{pL}\right) + 1 \qquad (56)$$

where L is the distance between the two walls 1 and 2, and F and a have the same values for the two walls. The results in this case are the same as those for the first-order analysis. This is, of course, because the velocity and temperature profiles as given by equation (53) and (54) are linear. The velocity results are in good agreement with experiment [1 (p. 721)].

FULLY DEVELOPED FLOW AND HEAT TRANSFER IN PASSAGES

For fully developed flow in a tube, the shear stress varies linearly with distance from the centerline, so that equation (22) becomes

$$\frac{r}{r_0}\tau_0 = -\mu\frac{dU_x}{dr}$$

or

$$U_x - U_0 = -\frac{\tau_0}{2\mu r_0}(r^2 - r_0^2) \qquad (57)$$

The derivatives in equation (25) can be calculated by setting $r^2 = z^2 + y^2$ in equation (57) and letting $y = 0$ after differentiation. Then the velocity of the gas at the wall is, with $U_w = 0$,

$$U_0 = \sqrt{\left(\frac{\pi}{2}\right)\frac{(2-F)}{F}\frac{\mu\sqrt{RT}}{p}\frac{\tau_0}{\mu}} +$$
$$+ \frac{27\pi}{32}\left(\frac{\mu\sqrt{RT}}{p}\right)^2\frac{\tau_0}{\mu r_0} \qquad (58)$$

The bulk or mixed mean velocity for flow in a tube is

$$U_b = \frac{\int_0^{r_0} U_x r\, dr}{\int_0^{r_0} r\, dr}$$

or, from equation (57),

$$U_b = U_0 + \frac{1}{4}\frac{\tau_0 r_0}{\mu} \qquad (59)$$

From equations (58) and (59),

$$\frac{\tau_0 r_0}{4\mu U_b} = \frac{1}{1 + 2\sqrt{(2\pi)}\frac{(2-F)}{F}\left(\frac{\mu\sqrt{RT}}{pr_0}\right) + \frac{27\pi}{8}\left(\frac{\mu\sqrt{RT}}{pr_0}\right)^2} \qquad (60)$$

A plot of equation (60) for $F = 1$ is given in Fig. 2. The term on the left-hand side of equation (60) is the same as the ratio of the actual pressure drop for the tube to that for continuum flow at the same velocity, if the pressure drop is small compared to the absolute pressure and entrance effects are small. These conditions are approximated in the data from reference 8 for hydrogen flow through a copper tube, and those data are included in Fig. 2 for comparison. These data are also representative of those for flow through glass tubes [8]. It is assumed that $F = 1$ for the data throughout the entire range of pressures, inasmuch as $F = 1$ in the free molecular region. Also included is the curve for first-order slip flow obtained
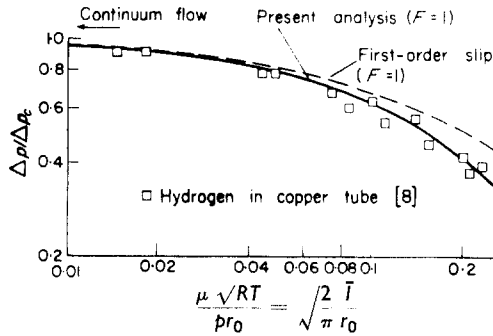
FIG. 2. Comparison of present analysis of fully developed second-order slip flow in tubes with first-order analysis and experimental data. (Diffuse reflection at wall.)

by neglecting the last term in the denominator

of the right-hand side of equation (60). The predicted curve for second-order slip flow appears to be in considerably better agreement with the data than does that for first-order slip, although there is some scatter in the data. When $\mu\sqrt{(RT)}/pr_0$ is on the order of 0·2, for which the analysis applies reasonably well, the difference between the first- and second-order equations is about 20 per cent. For values of $\mu\sqrt{(RT)}/pr_0$ greater than those shown, the predicted values begin to deviate considerably from the data, and a second-order slip flow analysis evidently is not applicable.

If flow between parallel plates is considered rather than flow through a tube, derivatives with respect to y are absent, and in place of equation (60) is

$$\frac{\tau_0 z_0}{3\mu U_b} = \frac{1}{1 + 3\sqrt{\left(\frac{\pi}{2}\right)\frac{(2-F)}{F}\left(\frac{\mu\sqrt{RT}}{pz_0}\right) + \frac{27\pi}{16}\left(\frac{\mu\sqrt{RT}}{pz_0}\right)^2}} \tag{61}$$

where z_0 is the half distance between the plates and

$$U_b = \frac{1}{z_0}\int_0^{z_0} U_x \, dz$$

Consider next the fully developed heat transfer in a tube with uniform wall heat flux. First-order slip flow for this case has been considered in reference 9. If axial conduction is neglected, the energy equation can be written for fully developed flow as

$$U_x \frac{\partial T}{\partial x} = a \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) \tag{62}$$

For uniform wall heat flux, $\partial T/\partial x$ is independent of r, and with the use of equation (57), equation (62) can be integrated to give

$$T - T_0 = \frac{\partial T/\partial x}{a}\left[\frac{U_0}{4}(r^2 - r_0^2) - \frac{\tau_0}{32\mu r_0}(r^4 - 4r_0^2 r^2 + 3r_0^4)\right] \tag{63}$$

The derivatives in the expression for the temperature jump at the wall [equation (52)] can be obtained by substituting $r^2 = z^2 + y^2$ in equation (63) and letting $y = 0$ after differentiation. Equation (52) then becomes

$$T_w - T_0 = \frac{\partial T/\partial x}{a}\left[\sqrt{(2\pi)}\frac{2-a}{a}\frac{\gamma}{(\gamma+1)Pr}\frac{\mu\sqrt{RT}}{p}\left(\frac{U_0 r_0}{2} + \frac{\tau_0 r_0^2}{8\mu}\right) + \frac{9\pi}{256}\frac{(177\gamma - 145)}{\gamma+1}\right.$$
$$\left. \left(\frac{\mu\sqrt{RT}}{p}\right)^2\left(\frac{3}{4}U_0 - \frac{1}{16}\frac{\tau_0 r_0}{\mu}\right)\right]. \tag{64}$$

The bulk or mixed mean temperature for flow in a tube is

$$T_b = \frac{\int_0^{r_0} T U_x r \, dr}{\int_0^{r_0} U_x r \, dr} \qquad (65)$$

With the use of equations (57) and (63),

$$T_b - T_0 =$$

$$\frac{\frac{\partial T}{\partial x} r_0^2 \left(-\frac{1}{2} U_0^2 - \frac{1}{3} \frac{\tau_0 U_0 r_0}{\mu} - \frac{11}{192} \frac{\tau_0^2 r_0^2}{\mu^2} \right)}{8a \left(\frac{1}{2} U_0 + \frac{1}{8} \frac{\tau_0 r_0}{\mu} \right)} \qquad (66)$$

or

$$T_w - T_b = T_w - T_0 +$$

$$\frac{\frac{\partial T}{\partial x} \tau_0 r_0^3 \left(\frac{11}{192} + \frac{1}{3} \frac{U_0 \mu}{\tau_0 r_0} + \frac{1}{2} \frac{U_0^2 \mu^2}{\tau_0^2 r_0^2} \right)}{8a\mu \left(\frac{1}{8} + \frac{1}{2} \frac{U_0 \mu}{\tau_0 r_0} \right)}. \qquad (67)$$

Writing a heat balance on a cylindrical element of fluid of radius r_0 gives

$$\frac{\partial T}{\partial x} = \frac{2q_0}{r_0 \rho U_b c_p} \qquad (68)$$

where q_0 is the heat transfer per unit area from the wall to the gas. Substituting equation (68) in (67) and using the definition for Nusselt number, $Nu = 2q_0 r_0/k(T_w - T_b)$, and equation (59) for U_b gives

$$\frac{1}{Nu} = \frac{(T_w - T_0) k}{2q_0 r_0} +$$

$$\frac{\frac{11}{48} \left[1 + \frac{64}{11} \frac{U_0 \mu}{\tau_0 r_0} + \frac{96}{11} \left(\frac{U_0 \mu}{\tau_0 r_0} \right)^2 \right]}{\left(1 + 4 \frac{U_0 \mu}{\tau_0 r_0} \right)^2}. \qquad (69)$$

The first term on the right-hand side of equation (69) is obtained from equations (64), (68) and (59) as

$$\frac{(T_w - T_0) k}{2q_0 r_0} = \sqrt{\left(\frac{\pi}{2} \right)} \frac{(2 - a)}{a} \frac{\gamma}{(\gamma + 1) Pr}$$

$$\frac{\mu \sqrt{RT}}{p r_0} - \frac{9\pi}{1024} \frac{(177\gamma - 145)}{\gamma + 1}$$

$$\frac{1 - 12 \frac{U_0 \mu}{\tau_0 r_0}}{1 + 4 \frac{U_0 \mu}{\tau_0 r_0}} \left(\frac{\mu \sqrt{RT}}{p r_0} \right)^2 \qquad (70)$$

and, from equation (58),

$$\frac{U_0 \mu}{\tau_0 r_0} = \sqrt{\left(\frac{\pi}{2} \right)} \frac{(2 - F) \mu \sqrt{RT}}{F} \frac{}{p r_0}$$

$$+ \frac{27\pi}{32} \left(\frac{\mu \sqrt{RT}}{p r_0} \right)^2. \qquad (71)$$

From equations (71), (70), and (69), Nusselt number can be calculated as a function of $\mu \sqrt{(RT)}/(p r_0)$. The ratio Nu/Nu_c, where $Nu_c = 48/11$, is plotted against $\mu \sqrt{(RT)}/(p r_0)$. in Fig. 3. Curves are shown for $\gamma = 1.4$, $Pr = 0.7$, $F = 1$, and for $a = 1$ and 0.5.
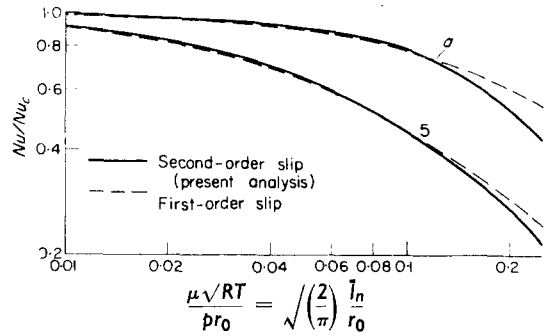

FIG. 3. Fully developed Nusselt number ratio for flow in a tube at uniform wall heat flux. $F = 1$, $\gamma = 1.4$, $Pr = 0.7$.

These values for γ and Prandtl number correspond approximately to air and most diatomic gases. Included for comparison are curves for first-order slip flow and temperature jump. Second-order effects in this case are somewhat less than those in Fig. 2; however, the differences between the first- and second-order equations are still on the order of 15 per cent at a value of $\mu \sqrt{(RT)}/(p r_0)$ of 0.2 and an a of 1. The differences are less for smaller values of a.

SUMMARY OF RESULTS

The effects of second-order normal and tangential derivatives on the velocity and temperature jumps at a wall in a rarefied gas were considered. Use was made of effective mean free paths for momentum and energy transfer that differ from the actual mean free path because of factors such as persistence of velocities, dependence of free path on velocity, etc. The

effective mean free paths were related to viscosities and other measurable quantities. The usual Navier–Stokes and energy equations in the gas was shown to be consistent with the use of second-order boundary conditions since, according to the analysis, the second-order terms are zero in the interior of the gas [equations (22) and (43)]. The velocity and temperature jumps at the walls are given by equations (22), (24), (22a), (24a), (51) and (52). The results agree with experiment at somewhat lower densities than does the usual first-order analysis.

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Résumé—Les effets des termes de second ordre sur les sauts de vitesse et de température à une paroi sont obtenus par une dérivation physique. L'analyse emploie les concepts des libres parcours moyens effectifs pour le transport de quantité de mouvement et d'énergie, on obtient les libres parcours moyens effectifs à partir des viscosités et des conductivités thermiques connues. L'analyse de l'écoulement de glissement du second ordre s'applique à des pressions bien plus basses que l'analyse du premier ordre et s'applique aussi bien à des gaz polyatomiques qu'à des gaz monoatomiques.

On considère plusieurs exemples illustratifs incluant l'écoulement entièrement développé et le transport de chaleur dans un tube. On a remarqué les différences de l'ordre de 20 pour cent entre les corrections du premier et du second ordre dans la région pour laquelle l'analyse semble s'appliquer.

Zusammenfassung—Die Einflüsse von Ausdrücken zweiter Ordnung auf Geschwindigkeits- und Temperatursprünge in Wandnähe liessen sich mit Hilfe einer physikalischen Ableitung erhalten. Die Analysis benützt das Konzept der effektiven mittleren freien Weglängen für den Impuls- und Energietransport; diese effektiven mittleren freien Weglängen wurden aus der bekannten Viskosität und Wärmeleitfähigkeit ermittelt. Die Gleitströmungsanalysis zweiter Ordnung ist für etwas geringere Drücke anwendbar als die Analysis erster Ordnung und gilt sowohl für einatomige als auch für mehratomige Gase. Verschiedene anschauliche Beispiele auch für voll ausgebildete Strömung und ausgebildeten Wärmeübergang im Rohr werden angegeben. Differenzen zwischen den Korrekturen erster und zweiter Ordnung wurden im Bereich in dem die Analysis anwendbar erscheint in der Grössenordnung von 20 Prozent festgestellt.

Аннотация—Влияние членов второго порядка на скачки скорости и температуры на стенке получено с помощью физического дифференцирования. При анализе используются понятия эффективных средних путей свободного пробега для переноса импульса и энергии; эффективные средние свободные пути пробега получены по известным значениям вязкостей и теплопроводностей. Анализ скользящего потока второго рода применяется при давлениях немного меньших, чем при анализе первого рода и применяется как к многоатомным, так и к одноатомным газам. Показано несколько примеров, причем рассмотрено полностью развитое течение и теплообмен в трубе. Отмечались разности между поправками первого и второго порядка, приблизительно около 20% в области, для которой применялся этот анализ.